DIMENSIONS OF SINGULARITY CATEGORIES OF HYPERSURFACES OF COUNTABLE REPRESENTATION TYPE

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Abstract. The Orlov spectrum and Rouquier dimension are invariants of a triangulated category to measure how big the category is, and they have been studied actively. In this paper, we investigate the singularity category $\mathrm{D_{sg}}(R)$ of a hypersurface R of countable representation type. For a thick subcategory $\mathcal{T}$ and a full subcategory $\mathcal{X}$ of $\mathcal{T}$, we calculate the Rouquier dimension of $\mathcal{T}$ with respect to $\mathcal{X}$. Furthermore, we prove that the level in $\mathrm{D_{sg}}(R)$ of the residue field of R with respect to each nonzero object is at most one.

1. Introduction

The Orlov spectrum and Rouquier dimension [5] measure how big a triangulated category is, but it is basically rather hard to calculate them. Thus, the notions of a level [4] and a relative Rouquier dimension [1] are introduced to measure how far one given object/subcategory from another given object/subcateogry. The main purpose of this paper is to report on these two invariants for the singularity category of a hypersurface of countable representation type.

Let k be an algebraically closed field of characteristic not two, and let R be a complete local hypersurface over k with countable representation type. Denote by $\mathrm{D_{sg}}(R)$ the singularity category of R. Let $\mathrm{D^o_{sg}}(R)$ be the full subcategory of $\mathrm{D_{sg}}(R)$ consisting of objects locally zero on the punctured spectrum of R. The main results of this paper are the following two theorems.

Theorem 1. *For all nonzero objects M of $\mathrm{D_{sg}}(R)$ one has*

$$\mathrm{level}^{M}_{\mathrm{D_{sg}}(R)}(k) \leq 1.$$

Theorem 2. *Let $\mathcal{T}$ be a nonzero thick subcategory of $\mathrm{D_{sg}}(R)$, and let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$. Then the following statements hold.*

(1) *$\mathcal{T}$ coincides with either $\mathrm{D_{sg}}(R)$ or $\mathrm{D^o_{sg}}(R)$.*

(2) (a) *If $\mathcal{T} = \mathrm{D_{sg}}(R)$, then*

$$\dim_{\mathcal{X}} \mathcal{T} = \begin{cases} 0 & (\text{if } \langle \mathcal{X} \rangle = \mathcal{T}), \\ 1 & (\text{if } \langle \mathcal{X} \rangle \neq \mathcal{T},\ \mathcal{X} \nsubseteq \mathrm{D^o_{sg}}(R)), \\ \infty & (\text{if } \mathcal{X} \subseteq \mathrm{D^o_{sg}}(R)). \end{cases}$$

The detailed version of this paper will be submitted for publication elsewhere.

(b) *If* $\mathcal{T} = \mathrm{D}^{\mathrm{o}}_{\mathrm{sg}}(R)$, *then*

$$\dim_{\mathcal{X}} \mathcal{T} = \begin{cases} 0 & (\textit{if } \langle \mathcal{X} \rangle = \mathcal{T}), \\ 1 & (\textit{if } \langle \mathcal{X} \rangle \neq \mathcal{T},\ \#\mathrm{ind}\mathcal{X} = \infty), \\ \infty & (\textit{if } \#\mathrm{ind}\mathcal{X} < \infty). \end{cases}$$

2. Preliminaries

We recall the definitions of several basic notions which are used in the later sections.

Definition 3. Let $\mathcal{T}$ be a triangulated category.

(1) For a subcategory $\mathcal{X}$ of $\mathcal{T}$ we denote by $\langle \mathcal{X} \rangle$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$ which is closed under isomorphisms, shifts, finite direct sums and direct summands.

(2) For subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{T}$ we denote by $\mathcal{X} * \mathcal{Y}$ the subcategory consisting of objects $M \in \mathcal{T}$ such that there is an exact triangle $X \to M \to Y \to X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Set $\mathcal{X} \diamond \mathcal{Y} := \langle \langle \mathcal{X} \rangle * \langle \mathcal{Y} \rangle \rangle$.

(3) For a subcategory $\mathcal{X}$ of $\mathcal{T}$ we put $\langle \mathcal{X} \rangle_0 := 0$, $\langle \mathcal{X} \rangle_1 := \langle \mathcal{X} \rangle$, and inductively define $\langle \mathcal{X} \rangle_n := \mathcal{X} \diamond \langle \mathcal{X} \rangle_{n-1}$ for $n \geq 2$. We set $\langle M \rangle_n := \langle \{M\} \rangle_n$ for an object $M \in \mathcal{T}$.

(4) The *generation time* of an object $M \in \mathcal{T}$ is defined by

$$\mathrm{gt}_{\mathcal{T}}(M) := \inf\{n \geq 0 \mid \mathcal{T} = \langle M \rangle_{n+1}\}.$$

If $\mathrm{gt}_{\mathcal{T}}(M)$ is finite, M is called a *strong generator* of $\mathcal{T}$.

(5) The *Orlov spectrum* and *(Rouquier) dimension* of $\mathcal{T}$ are defined as follows.

$$\mathrm{OSpec}(\mathcal{T}) := \{\mathrm{gt}_{\mathcal{T}}(M) \mid M \text{ is a strong generator of } \mathcal{T}\},$$
$$\dim \mathcal{T} := \inf \mathrm{OSpec}(\mathcal{T}) = \inf\{n \geq 0 \mid \mathcal{T} = \langle M \rangle_{n+1} \text{ for some } M \in \mathcal{T}\}.$$

(6) Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$. The *dimension* of $\mathcal{T}$ *with respect to* $\mathcal{X}$ is defined by

$$\dim_{\mathcal{X}} \mathcal{T} := \inf\{n \geq 0 \mid \mathcal{T} = \langle \mathcal{X} \rangle_{n+1}\}.$$

(7) Let M, N be objects of $\mathcal{T}$. Then the *level* of N with respect to M is defined by

$$\mathrm{level}^M_{\mathcal{T}}(N) := \inf\{n \geq 0 \mid N \in \langle M \rangle_{n+1}\}.$$

Definition 4. Let R be a Noetherian ring.

(1) We denote by $\mathrm{D}^{\mathrm{b}}(R)$ the bounded derived category of finitely generated R-modules.

(2) A *perfect* complex is by definition a bounded complex of finitely generated projective modules.

(3) We denote by $\mathrm{perf}(R)$ the subcategory of $\mathrm{D}^{\mathrm{b}}(R)$ consisting of complexes quasi-isomorphic to perfect complexes.

(4) The *singularity category* of R is defined by

$$\mathrm{D}_{\mathrm{sg}}(R) := \mathrm{D}^{\mathrm{b}}(R)/\mathrm{perf}(R),$$

that is, the Verdier quotient of $\mathrm{D}^{\mathrm{b}}(R)$ by $\mathrm{perf}(R)$.

Let R be a Cohen–Macaulay local ring. Let $\mathrm{CM}(R)$ be the category of maximal Cohen-Macaulay R-modules, and $\underline{\mathrm{CM}}(R)$ the stable category of $\mathrm{CM}(R)$. The following theorem is celebrated and fundamental; see [7, Theorem 4.4.1].

Theorem 5 (Buchweitz). *Let R be a Gorenstein local ring of Krull dimension d. Then $\underline{\mathrm{CM}}(R)$ has the structure of a triangulated category, and there exist mutually inverse triangle equivalence functors*

$$F : \mathrm{D_{sg}}(R) \rightleftarrows \underline{\mathrm{CM}}(R) : G,$$

such that $GM = M$ for each maximal Cohen–Macaulay R-module M and $FN = \Omega^d N[d]$ for each finitely generated R-module N.

By virtue of Theorem 5, for a Gorenstein local ring, the study of generation in the singularity category reduces to the stable category of maximal Cohen–Macaulay modules.

3. The relationship between the singularity categories of R and $R^\sharp$

Let $(R, \mathfrak{m}, k)$ be a complete equicharacteristic local hypersurface of (Krull) dimension d. Then thanks to Cohen's structure theorem we can identify R with a quotient of a formal power series ring over k:

$$R = k[\![x_0, x_1, \ldots, x_d]\!]/(f)$$

with $0 \neq f \in (x_0, x_1, \ldots, x_d)^2$. We define a hypersurface of dimension $d+1$:

$$R^\sharp = k[\![x_0, x_1, \ldots, x_d, y]\!]/(f + y^2).$$

Note that the element y is $R^\sharp$-regular and there is an isomorphism $R^\sharp/yR^\sharp \cong R$. The main purpose of this section is to compare generation in the singularity categories $\mathrm{D_{sg}}(R)$ and $\mathrm{D_{sg}}(R^\sharp)$. As both R and $R^\sharp$ are Gorenstein, in view of Theorem 5 and the remark following the theorem, it suffices to investigate the stable categories of maximal Cohen–Macaulay modules $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R^\sharp)$.

The following result is a consequence of [14, Proposition 12.4], which plays a key role to compare generation in $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R^\sharp)$.

Lemma 6. *The assignments $M \mapsto \Omega_{R^\sharp} M$ and $N \mapsto N/yN$ define triangle functors $\Phi : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R^\sharp)$ and $\Psi : \underline{\mathrm{CM}}(R^\sharp) \to \underline{\mathrm{CM}}(R)$ satisfying*

$$\Psi\Phi(M) \cong M \oplus M[1], \qquad \Phi\Psi(N) \cong N \oplus N[1].$$

In particular, Φ and Ψ are both equivalences up to direct summands.

Applying this lemma, we deduce relationships of levels in $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R^\sharp)$.

Proposition 7. *One has the following equalities.*

(1) $\mathrm{level}^{M}_{\underline{\mathrm{CM}}(R)}(\Omega^d_R k) = \mathrm{level}^{\Omega_{R^\sharp} M}_{\underline{\mathrm{CM}}(R^\sharp)}(\Omega^{d+1}_{R^\sharp} k)$ *for each $M \in \underline{\mathrm{CM}}(R)$.*

(2) $\mathrm{level}^{N}_{\underline{\mathrm{CM}}(R^\sharp)}(\Omega^{d+1}_{R^\sharp} k) = \mathrm{level}^{N/yN}_{\underline{\mathrm{CM}}(R)}(\Omega^d_R k)$ *for each $N \in \underline{\mathrm{CM}}(R^\sharp)$.*

Using Lemma 6 again, we get relationships of generation times in $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R^\sharp)$.

Proposition 8. *The following statements holds true.*

(1) *If $M \in \underline{\mathrm{CM}}(R)$ is a strong generator, then so is $\Omega_{R^\sharp} M \in \underline{\mathrm{CM}}(R^\sharp)$, and $\mathrm{gt}_{\underline{\mathrm{CM}}(R)}(M) = \mathrm{gt}_{\underline{\mathrm{CM}}(R^\sharp)}(\Omega_{R^\sharp} M)$.*

(2) *If $N \in \underline{\mathrm{CM}}(R^\sharp)$ is a strong generator, then so is $N/yN \in \underline{\mathrm{CM}}(R)$, and $\mathrm{gt}_{\underline{\mathrm{CM}}(R^\sharp)}(N) = \mathrm{gt}_{\underline{\mathrm{CM}}(R)}(N/yN)$.*

4. The singularity category of a hypersurface of countable representation type

In this section, we prove our main results, that is, Theorems 10 and 11 from the Introduction. We start by the following general lemma.

Lemma 9. *Let $\mathcal{T}$ be a triangulated category. Let*

$$X \xrightarrow{\binom{f_1}{f_2}} Y_1 \oplus M \xrightarrow{(g_1,\alpha)} N \rightsquigarrow, \qquad M \xrightarrow{\binom{\alpha}{g_2}} N \oplus Y_2 \xrightarrow{(h_1,h_2)} Z \rightsquigarrow$$

be exact triangles in $\mathcal{T}$. Then the sequence

$$X \xrightarrow{\binom{f_1}{g_2 f_2}} Y_1 \oplus Y_2 \xrightarrow{(h_1 g_1, -h_2)} Z \rightsquigarrow$$

is also an exact triangle in $\mathcal{T}$.

Let $(R, \mathfrak{m}, k)$ be a complete equicharacteristic local hypersurface of dimension d. Assume that k has characteristic different from two, and that R has *countable (Cohen–Macaulay) representation type*, namely, there exist infinitely but only countably many isomorphism classes of indecomposable maximal Cohen–Macaulay R-modules. Then f is either of the following; see [10, (1.6)].

$$(\mathrm{A}_\infty): \ x_0^2 + x_2^2 + \cdots + x_d^2,$$
$$(\mathrm{D}_\infty): \ x_0^2 x_1 + x_2^2 + \cdots + x_d^2.$$

In this case, all objects in $\mathrm{CM}(R)$ are completely classified [6, 8, 11].

Now we can state and prove the following result regarding levels in $\underline{\mathrm{CM}}(R)$.

Theorem 10. *Let k be an algebraically closed field of characteristic not two. Let R be a d-dimensional complete local hypersurface over k of countable representation type. Then*

$$\Omega_R^d k \in \langle M \rangle_2^{\underline{\mathrm{CM}}(R)}$$

for all nonzero objects $M \in \underline{\mathrm{CM}}(R)$. In other words, $\mathrm{level}_{\underline{\mathrm{CM}}(R)}^M(\Omega_R^d k) \leq 1$.

Proof. Proposition 7(2) reduces to the case $d = 1$. Thus we have the two cases:

$$(1)\ R = k[\![x,y]\!]/(x^2), \qquad (2)\ R = k[\![x,y]\!]/(x^2 y).$$

(1): Thanks to [6, 4.1], the indecomposable objects of $\underline{\mathrm{CM}}(R)$ are the ideals $I_n = (x, y^n)$ with $n \in \mathbb{Z}_{>0} \cup \{\infty\}$, where $I_\infty := (x)$. By [12, 6.1] there exist exact triangles

$$I_n \to I_{n-1} \oplus I_{n+1} \to I_n \rightsquigarrow \quad (n \in \mathbb{Z}_{>0}),$$

where $I_0 := 0$. Applying Lemma 9, we obtain exact triagles

$$I_n \to I_1 \oplus I_{2n-1} \to I_n \rightsquigarrow \quad (n \in \mathbb{Z}_{>0}),$$

and from [3, Proposition 2.1] we obtain an exact triangle $I_\infty \to I_1 \to I_\infty \rightsquigarrow$. It is observed from these triangles that $\Omega k = I_1$ is in $\langle M \rangle_2$ for each nonzero object $M \in \underline{\mathrm{CM}}(R)$.

(2): Similarly. □

Proof of Theorem 1. The assertion is immediate from Theorems 10 and 5. □

Recall that a subcategory of a triangulated category is called *thick* if it is a triangulated subcategory closed under direct summands. We denote by $\underline{\mathrm{CM}}^{\mathrm{o}}(R)$ the subcategory of $\underline{\mathrm{CM}}(R)$ consisting of maximal Cohen–Macaulay R-modules that are locally free on the punctured spectrum of R. The category $\underline{\mathrm{CM}}^{\mathrm{o}}(R)$ is a thick subcategory of $\underline{\mathrm{CM}}(R)$, and in particular it is a triangulated category. For an essentially small additive category $\mathcal{C}$ we denote by $\mathrm{ind}\mathcal{C}$ the set of nonisomorphic indecomposable objects of $\mathcal{C}$. We can now state and prove the following result concerning relative Rouquier dimensions in $\underline{\mathrm{CM}}(R)$.

Theorem 11. *Let k be an algebraically closed field of characteristic not two. Let R be a d-dimensional complete local hypersurface over k of countable representation type. Let $\mathcal{T} \neq 0$ be a thick subcategory of $\underline{\mathrm{CM}}(R)$, and let $\mathcal{X}$ be a subcategory of $\mathcal{T}$. Then:*

(1) *$\mathcal{T}$ coincides with either $\underline{\mathrm{CM}}(R)$ or $\underline{\mathrm{CM}}^{\mathrm{o}}(R)$.*

(2) (a) *When $\mathcal{T} = \underline{\mathrm{CM}}(R)$, one has*

$$\dim_{\mathcal{X}} \mathcal{T} = \begin{cases} 0 & (\text{if } \langle \mathcal{X} \rangle = \mathcal{T}), \\ 1 & (\text{if } \langle \mathcal{X} \rangle \neq \mathcal{T} \text{ and } \mathcal{X} \not\subseteq \underline{\mathrm{CM}}^{\mathrm{o}}(R)), \\ \infty & (\text{if } \mathcal{X} \subseteq \underline{\mathrm{CM}}^{\mathrm{o}}(R)). \end{cases}$$

(b) *When $\mathcal{T} = \underline{\mathrm{CM}}^{\mathrm{o}}(R)$, one has*

$$\dim_{\mathcal{X}} \mathcal{T} = \begin{cases} 0 & (\text{if } \langle \mathcal{X} \rangle = \mathcal{T}), \\ 1 & (\text{if } \langle \mathcal{X} \rangle \neq \mathcal{T} \text{ and } \#\mathrm{ind}\mathcal{X} = \infty), \\ \infty & (\text{if } \#\mathrm{ind}\mathcal{X} < \infty). \end{cases}$$

Proof. (1) We combine [13, Theorem 6.8] and [3, Theorem 1.1]. The singular locus of R consists of two points $\mathfrak{p}$ and $\mathfrak{m}$, and its specialization-closed subsets are $\mathrm{V}(\mathfrak{p})$, $\mathrm{V}(\mathfrak{m})$ and $\emptyset$. These correspond to the thick subcategories $\underline{\mathrm{CM}}(R)$, $\underline{\mathrm{CM}}^{\mathrm{o}}(R)$ and 0.

(2) Part (a) follows from [3, Theorem 1.1]. Let us show part (b). When $\#\mathrm{ind}\mathcal{X} < \infty$, let $X_1, \dots, X_n$ be all the indecomposable objects in $\mathcal{X}$. Suppose that $\dim_{\mathcal{X}} \underline{\mathrm{CM}}^{\mathrm{o}}(R)$ is finite, say m. Then it follows that $\underline{\mathrm{CM}}^{\mathrm{o}}(R) = \langle \mathcal{X} \rangle_{m+1} = \langle X \rangle_{m+1}$, where $X := X_1 \oplus \cdots \oplus X_n \in \underline{\mathrm{CM}}^{\mathrm{o}}(R)$. Hence $\underline{\mathrm{CM}}^{\mathrm{o}}(R)$ has finite Rouquier dimension. By [9, Theorem 1.1(2)], the local ring R has to have at most an isolated singularity. However, in either case of the types (A_∞) and (D_∞) we see that the nonmaximal prime ideal $(x_0, x_2, \dots, x_d)R$ belongs to the singular locus of R, which is a contradiction. Consequently, we obtain $\dim_{\mathcal{X}} \underline{\mathrm{CM}}^{\mathrm{o}}(R) = \infty$.

From now on we consider the case where $\langle \mathcal{X} \rangle \neq \mathcal{T} = \underline{\mathrm{CM}}^{\mathrm{o}}(R)$ and $\#\mathrm{ind}\mathcal{X} = \infty$. We adopt the same notation as in the proof of Theorem 10.

Assume that R has type (A_∞). As $\langle \mathcal{X} \rangle$ is a proper subcategory, we can find a positive integer n such that $I_n \notin \mathcal{X}$. Since there are infinitely many indecomposable objects in $\mathcal{X}$, we can also find an integer $m > n$ such that $I_m \in \mathcal{X}$. There exists an exact triangle

$$I_m \to I_n \oplus I_{2m-n} \to I_m \rightsquigarrow$$

in $\mathcal{T} = \underline{\mathrm{CM}}^{\mathrm{o}}(R)$, which shows that I_n belongs to $\langle \mathcal{X} \rangle_2$. Therefore, we get $\dim_{\mathcal{X}} \mathcal{T} \leq 1$. Since $\langle \mathcal{X} \rangle \neq \mathcal{T}$, we have $\dim_{\mathcal{X}} \mathcal{T} \neq 0$. Consequently, we obtain $\dim_{\mathcal{X}} \mathcal{T} = 1$.

Suppose that R is of type (D_∞). Similarly as the case of type (A_∞). □

Proof of Theorem 2. Theorems 11 and 5 immediately deduce the assertion. □

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